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SIGN CHANGES OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF INTEGRAL WEIGHT, 2

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Abstract. In this paper, we investigate the sign changes of Fourier coefficients of half-integral weight Hecke eigenforms and give two quantitative results on the number of sign changes.

1. Introduction

The study of sign-changes of Fourier coefficients of automorphic forms is recently very active. For modular (Hecke eigen-)forms of integral weight, the consequential result from Matomäki and Radziwill [14] is exceptionally charming, where the multiplicative properties of the Fourier coefficients play a substantial role. However the modular forms of half-integral weight do not share the same kind of multiplicativity, and many problems deserve delving.

Let $\ell \geqslant 2$ be a positive integer, and denote by $\mathfrak{S}_{\ell+1/2}$ the set of all cusp forms of weight $\ell + 1/2$ for the congruence subgroup $\Gamma_0(4)$. Consider the coefficients in the Fourier expansion of a complete Hecke eigenform $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ at ∞,

$$\mathfrak{f}(z) = \sum_{n\geqslant 1} \lambda_{\mathfrak{f}}(n) n^{\ell/2-1/4} \mathrm{e}(nz) \quad (z \in \mathscr{H}), \tag{1.1}$$

where $\mathrm{e}(z) = \mathrm{e}^{2\pi \mathrm{i} z}$ and $\mathscr{H}$ is the Poincaré upper half plane. A specific question is the number of sign-changes when all $\lambda_{\mathfrak{f}}(n)$ are real. We interlude with the meaning of sign-changes of a sequence.

Let $\mathcal{N}$ be a subset of $\mathbb{N}$ endowed with the ordering of integers. The sets of squarefree integers or arithmetic progressions are basic examples. Given a real sequence $\{a_n\}_{n\in\mathcal{N}}$. A sign-change is realized via a closed and bounded interval $[i, j] \subset (0, \infty)$ such that

(i) its end-points i, j lie in $\mathcal{N}$ and satisfy $a_i a_j < 0$, and
(ii) $a_n = 0$ for all $n \in (i, j) \cap \mathcal{N}$.

The sequence $\{a_n\}_{n\in\mathcal{N}}$ is said to have a sign-change in the interval I if I contains one such interval $[i, j]$. Besides, the number of sign-changes of $\{a_n\}_{n\in\mathcal{N}}$ in $[1, x]$, denoted by $\mathcal{C}^{\mathcal{N}}(x)$, is meant to be the number of intervals $[i, j]$ contained in $[1, x]$.[†]

Let $\flat$ be the set of squarefree numbers. Hulse, Kiral, Kuan & Lim [6] proved that the sequence $\{\lambda_{\mathfrak{f}}(t)\}_{t\in\flat}$ has an infinity of sign-changes. A quantitative version is given in Lau, Royer & Wu [13, Theorem 4], which says $\mathcal{C}_{\mathfrak{f}}^{\flat}(x) \gg x^{(1-4\varrho)/5-\varepsilon}$ where $\mathcal{C}_{\mathfrak{f}}^{\flat}(x)$ denotes the number of sign-changes of $\{\lambda_{\mathfrak{f}}(t)\}_{n\in\flat}$ in $[1, x]$ and the constant ϱ is determined by (3.5) below. Conjecturally $\varrho = \varepsilon$ but it is still hard to guess the tight lower bound.

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[†]An equivalent but slightly different formulation is given in [13].

On the other hand, Meher & Murty [15] studied the sign-change problem for Hecke eigenforms $\mathfrak{f}$ in Kohnen plus subspace of $\mathfrak{S}_{\ell+1/2}$. A form $\mathfrak{f}$ in the plus space has its Fourier coefficients supported at integers $n \equiv 0$ or $(-1)^\ell \,(\bmod 4)$, i.e. $\mathfrak{f}$ has the Fourier expansion at ∞ of the form

$$\mathfrak{f}(z) = \sum_{(-1)^\ell n \equiv 0,1 (\bmod 4)} \lambda_{\mathfrak{f}}(n) n^{\ell/2-1/4} \mathrm{e}^{2\pi i n z}.$$

When $\mathfrak{f}$ is a Hecke eigenform in the plus space and its coefficients $\lambda_f(n)$ are all real, Meher & Murty proved in [15, Theorem 2] that $\{\lambda_f(n)\}_{n\in\mathbb{N}}$ has a sign-change in the short interval $(x, x + x^{43/70+\varepsilon}]$ for any $\varepsilon > 0$ and for all sufficiently large $x \geqslant x_0(\varepsilon)$. An immediate consequence is $\mathcal{C}_{\mathfrak{f}}^{\mathbb{N}}(x) \gg x^{27/70-\varepsilon}$. This work naturally motivates the sign-change problem for arithmetic progressions.

In this paper, we furnish progress, based on our work in [10], in the above problems for complete Hecke eigenforms $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$. Firstly for the case $\mathcal{N} = \flat$, we sharpen the lower bound for $\mathcal{C}_{\mathfrak{f}}^{\flat}(x)$.

Theorem 1. *Let $\ell \geqslant 2$ be an integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ a complete Hecke eigenform such that its Fourier coefficients are real. Let ϱ be defined as in* (3.5) *below, and ϑ any number satisfying*

$$0 < \vartheta < \min(\tfrac{1-2\varrho}{3}, \tfrac{1}{4}).$$

Then

$$\mathcal{C}_{\mathfrak{f}}^{\flat}(x) \gg_{\mathfrak{f},\vartheta} x^{\vartheta} \tag{1.2}$$

for all $x \geqslant x_0(\mathfrak{f}, \vartheta)$, where the constant $x_0(\mathfrak{f}, \vartheta)$ and the implied constant depend on $\mathfrak{f}$ and ϑ only.

Remark 1. In particular, Conrey & Iwaniec [2] gives $\varrho = \frac{1}{6} + \varepsilon$ which leads to

$$\mathcal{C}_{\mathfrak{f}}^{\flat}(x) \gg_{\mathfrak{f},\varepsilon} x^{2/9-\varepsilon}$$

for all $x \geqslant x_0(\mathfrak{f}, \varepsilon)$, improving the exponent $\frac{1}{15} - \varepsilon$ in [13].

Secondly we generalize the case of $\mathcal{N} = \mathbb{N}$ in Meher & Murty [15] to arithmetic progressions. Let $Q \geqslant 1$ be an integer, and $a = 0$ or $a \in \mathbb{N}$ with $(a, Q) = 1$. Define

$$\mathcal{A} = \mathcal{A}_{a,Q} := \{n \in \mathbb{N} : \ n \equiv a \,(\bmod Q)\}. \tag{1.3}$$

We study the sign-changes of $\{\lambda_{\mathfrak{f}}(n)\}_{n\in\mathcal{A}}$ and sharpen the exponent $\frac{43}{70} + \varepsilon$ of Meher & Murty's result to $\frac{1}{2}$, which in turn gives the better lower bound $\mathcal{C}_{\mathfrak{f}}^{\mathbb{N}}(x) \gg x^{1/2}$.

Theorem 2. *Assume the same conditions for $\mathfrak{f}$ and ϱ in Theorem 1. Let $Q \geqslant 1$ be odd and $\mathcal{A} = \mathcal{A}_{a,Q}$ defined as in* (1.3). *Suppose one of the following condition holds:*

1° *$Q = 1$;*
2° *$a = 0$ and $Q = \prod_{p|Q} p^{\alpha_p}$ where all α_p are odd;*
3° *$(a, Q) = 1$ and $Q = \prod_{p|Q} p^{\alpha_p}$ where all α_p are $\geqslant 2$.*

Then there are positive constants $c_0 = c_0(\mathfrak{f}, Q)$ and $x_0 = x_0(\mathfrak{f}, Q)$ such that the sequence $\{\lambda_{\mathfrak{f}}(n)\}_{n\in\mathcal{A}}$ has at least one sign change in the interval $(x, x + c_0 x^{1/2}]$ for all $x \geqslant x_0$. In particular, we have

$$\mathcal{C}_{\mathfrak{f}}^{\mathcal{A}}(x) \gg_{\mathfrak{f},Q} x^{1/2}$$

for all $x \geqslant x_0$.

2. Methodologies

Let $\lambda_{\mathfrak{f}}(n)$ be the coefficients as in (1.1) and $\mathcal{N}$ a subset of $\mathbb{N}$. Define

$$S_{\mathfrak{f}}^{\mathcal{N}}(x) := \sum_{\substack{n\leqslant x\\ n\in\mathcal{N}}} \lambda_{\mathfrak{f}}(n). \tag{2.1}$$

A typical approach for the sign-change detection exploits the oscillation exhibited in the mean $S_{\mathfrak{f}}^{\mathcal{N}}(x)$, while to locate the sign-change, the mean over short intervals, i.e. $S_{\mathfrak{f}}^{\mathcal{N}}(x+h) - S_{\mathfrak{f}}^{\mathcal{N}}(x)$ for small h, will be a good device. Suppose a sign-change is found in the interval $[x, x+h]$ for every x large enough. Then it follows immediately that the number of sign-changes in $[1, x]$ is at least $x/h + O(1)$ (and hence $\gg x/h$). A standard way to study $S_{\mathfrak{f}}^{\mathcal{N}}(x)$ is via the Dirichlet series. But for various $\mathcal{N}$, we get different degree of its analytic information.

For $\mathcal{N} = \flat$, i.e. the case of squarefree integers, we only get an analytic continuation of the Dirichlet series

$$L_{\mathfrak{f}}^{\flat}(s) := \sum_{t\geqslant 1}^{\flat} \lambda_{\mathfrak{f}}(t) t^{-s} \tag{2.2}$$

in the half-plane $\Re e\, s > \frac{1}{2}$, where $\sum_{t\geqslant 1}^{\flat}$ ranges over squarefree integers $t \geqslant 1$. As illustrated in [13], it turns out that the weighted mean is more effective. Thus, to prove Theorem 1, we first derive (2.3) below,

$$\sum_{x\leqslant t\leqslant x+h}^{\flat} \lambda_{\mathfrak{f}}(t) \min\left\{\log\left(\frac{x+h}{t}\right), \log\left(\frac{x}{t}\right)\right\} \ll_{\varepsilon} h^{\frac{1}{2}} x^{\varepsilon}. \tag{2.3}$$

The better exponent $\frac{1}{2}$ (versus $\frac{3}{4}$ in [13]) of h is a key for the improvement. Another key is to have a mean square formula with better O-term. In [13], we showed that

$$\sum_{X<n\leqslant 2X} |\lambda_{\mathfrak{f}}(n)|^2 = D_{\mathfrak{f}} X + O_{\mathfrak{f},\varepsilon}\big(X^{\beta+\varepsilon}\big).$$

with $\beta = \frac{3}{4} + \varrho$. Here we sharpen it to $\beta = \frac{3}{4}$ in Lemma 4.1 and then conclude Theorem 1 with argument in [13]. This will be done in Section 4.

Next for $\mathcal{N} = \mathcal{A}$ (see (1.3)), we shall provide a truncated Voronoi formula for $S_{\mathfrak{f}}^{\mathcal{A}}(x)$ in Section 6. This result is itself interesting since the Voronoi formula is an vital tool for many applications, see [7], [11] for example. Then we complete the proof of Theorem 2 with the method of Heath-Brown and Tsang [5]. However the congruence condition underlying $\mathcal{A}$ gives rise to new (but interesting) difficulties. To transform the congruence, additive characters of modulus $d|Q$ will be invoked and then two consequences follow: the summands in the Voronoi formula are intertwined with Kloosterman-Salié sums, and the frequencies in the cosines are of the form $\sqrt{n}/d$. We need to select a suitable frequency for amplification with a pair of non-vanishing Salié sum and Fourier coefficient in the associated summand. The implementation is successful when Q fulfills the conditions in Theorem 2, which will be elucidated in Sections 7 & 8. It is worthwhile to remark that the mean square result of $\lambda_{\mathfrak{f}}(n)$ is not needed for the method in [5].

3. Background

A cusp form $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ has Fourier expansions at the three inequivalent cusps $\infty, -\frac{1}{2}, 0$ of $\Gamma_0(4)$, which are respectively given by (1.1), and (3.1), (3.2) below:

$$\begin{aligned} \mathfrak{g}(z) &:= 2^{\ell+1/2}(-8z+1)^{-(\ell+1/2)}\mathfrak{f}\left(\frac{4z}{-8z+1}\right) \\ &= 2^{\ell+1/2}\sum_{n\geqslant 1}\lambda_{\mathfrak{g}}(n)n^{\ell/2-1/4}\mathrm{e}(nz) \end{aligned} \tag{3.1}$$

and

$$\mathfrak{h}(z) := (-\mathrm{i}2z)^{-(\ell+1/2)}\mathfrak{f}\left(\frac{-1}{4z}\right) = \sum_{n\geqslant 1}\lambda_{\mathfrak{h}}(n)n^{\ell/2-1/4}\mathrm{e}(nz). \tag{3.2}$$

Following the argument in [13, Section 2.2], we have

$$\sum_{n\leqslant x}|\lambda_f(n)|^2 \sim x \qquad \text{(for all three cases } f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}). \tag{3.3}$$

When $\mathfrak{f}$ is a complete Hecke eigenform, we know from [10] that $\mathfrak{g}$ and $\mathfrak{h}$ are Hecke eigenforms of $\mathsf{T}(p^2)$ for all odd prime p. A consequence is, cf. [10, Lemma 3.2 with $\mathcal{Q} = \{2\}$]: for all odd $m \geqslant 1$, all squarefree t and $j \geqslant 0$,

$$\lambda_f(2^j t) = 0 \ \Rightarrow\ \lambda_f(2^j tm^2) = 0 \quad (f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}). \tag{3.4}$$

In addition, we have the following pointwise estimate, see [10, Lemma 3.3].

Lemma 3.1. *Let $\mathfrak{f}$ be a complete Hecke eigenform, $\mathfrak{g}$ and $\mathfrak{h}$ be defined as above. For any integer $m = tr^2$ where $t \geqslant 1$ is squarefree, we have*

$$\lambda_f(m) \ll_{\mathfrak{f}} |\lambda_f(t)|\tau(r)^2 + |\lambda_{\mathfrak{f}}(t)|\tau(r)^2 \ll_{\mathfrak{f},\varrho} t^{\varrho}\tau(r)^2$$

for $f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ respectively, where $\tau(n)$ is the divisor function and ϱ satisfies (3.5) below. The first implied $\ll$-constant depends only $\mathfrak{f}$ and the second implied $\ll$-constant depends at most on $\mathfrak{f}$ and ϱ.

Here ϱ denotes the exponent for which

$$\lambda_{\mathfrak{f}}(t) \ll_{\varrho} t^{\varrho} \qquad \forall\ t \text{ squarefree}, \tag{3.5}$$

i.e. the bound towards the Ramaujan Conjecture for the half-integral weight Hecke eigenforms. The conjectural value is $\varrho = \varepsilon$. Conrey & Iwaniec [2] obtained $\varrho = \frac{1}{6} + \varepsilon$.

Let $d \geqslant 1$ be an integer and $(u, d) = 1$. Define the twisted L-function for $\mathfrak{f}$ by

$$L_{\mathfrak{f}}(s, u/d) = \sum_{m\geqslant 1}\frac{\lambda_{\mathfrak{f}}(m)\mathrm{e}(mu/d)}{m^s} \qquad (\Re e\, s > 1) \tag{3.6}$$

and define similarly for $\mathfrak{g}$ and $\mathfrak{h}$. These twisted L-functions when attached with suitable factors may be expressed as integrals of $\mathfrak{f}$ along vertical geodesics, and extend to entire functions, cf. [6, (4.4)-(4.5)]. Moreover Hulse et al found the functional equation for $L_{\mathfrak{f}}(s, u/d)$, which is put in the following form

$$q_d^s L_\infty(s)L_{\mathfrak{f}}(s, u/d) = \mathrm{i}^{-(\ell+1/2)}q_d^{1-s}L_\infty(1-s)\widetilde{L}_{\mathfrak{f}}(1-s, v/d), \tag{3.7}$$

where $uv \equiv 1 \pmod d$ and $L_\infty(s) := (2\pi)^{-s}\Gamma\big(s + \frac{\ell}{2} - \frac{1}{4}\big)$ is the gamma factor, cf. [6, Lemma 4.3] and [10]. The conductor q_d and the dual L-function $\widetilde{L}_{\mathfrak{f}}(s, v/d)$ are defined as follows:

$$q_d = d \quad \text{or} \quad 2d \quad \text{according to } 4 \mid d \text{ or not,} \tag{3.8}$$

and

$$\widetilde{L}_{\mathfrak{f}}(s, v/d) := \sum_{n \geqslant 1} \lambda(n; d)\varpi_d(n, v)n^{-s}, \tag{3.9}$$

where

(3.10)

	$\lambda(n;d)$	$\varpi_d(n,v)$
$4 \mid d$	$\lambda_{\mathfrak{f}}(n)$	$\varepsilon_v^{2\ell+1}\big(\frac{d}{v}\big)\mathrm{e}\big(\frac{-nv}{d}\big)$
$2 \parallel d$	$\lambda_{\mathfrak{g}}(n)$	$\varepsilon_v^{2\ell+1}\big(\frac{d}{v}\big)\mathrm{e}\big(\frac{-nv}{4d}\big)$
$2 \nmid d$	$\lambda_{\mathfrak{h}}(n)$	$\mathrm{i}^{\ell+1/2}\varepsilon_d^{-(2\ell+1)}\big(\frac{v}{d}\big)\mathrm{e}\big(\frac{-\overline{4}nv}{d}\big)$

with $4\overline{4} \equiv 1 \pmod d$.

In [6], Hulse et al applied $L_{\mathfrak{f}}(s, u/d)$ to obtain the analytic properties of $L_{\mathfrak{f}}^{\flat}(s)$, which was sharpened to the following result [10, Theorem 1].

Lemma 3.2. *For a complete Hecke eigenform $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$, the series $L_{\mathfrak{f}}^{\flat}(s)$ extends analytically to a holomorphic function on $\Re e\, s > \frac{1}{2}$, and for any $\varepsilon > 0$,*

$$L_{\mathfrak{f}}^{\flat}(s) \ll_{\mathfrak{f},\varepsilon} (|\tau| + 1)^{1-\sigma+2\varepsilon} \qquad (\tfrac{1}{2} + \varepsilon \leqslant \sigma \leqslant 1 + \varepsilon, \tau \in \mathbb{R}), \tag{3.11}$$

where the implied constant depends on $\mathfrak{f}$ and ε only.

Remark 2. Using Lemma 3.2 in place of [13, Proposition 7], the estimate in (2.3) follows plainly from the same argument as in [13, Section 4.1], so we do not repeat here.

4. Proof of Theorem 1

We start with the following lemma where the O-term in (4.1) is smaller than [13, (14)].

Lemma 4.1. *Let $\ell \geqslant 2$ be a positive integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be a complete Hecke eigenform. Then for any $\varepsilon > 0$ and all $x \geqslant 2$, we have*

$$\sum_{n \leqslant x} |\lambda_{\mathfrak{f}}(n)|^2 = D_{\mathfrak{f}}\, x + O_{\mathfrak{f},\varepsilon}\big(x^{3/4+\varepsilon}\big), \tag{4.1}$$

where $D_{\mathfrak{f}}$ is a positive constant depending on $\mathfrak{f}$.

Proof. We choose two smooth compactly supported functions $w_\pm$ such that

- $w_-(x) = 1$ for $x \in [X + Y, 2X - Y], w_-(x) = 0$ for $x \geqslant 2X$ and $x \leqslant X$;
- $w_+(x) = 1$ for $x \in [X, 2X], w_+(x) = 0$ for $x \geqslant 2X + Y$ and $x \leqslant X - Y$;
- $w_\pm^{(j)}(x) \ll_j Y^{-j}$ for all $j \geqslant 0$;

- the Mellin transform of $w(x)$ is

$$\begin{aligned}\widehat{w_\pm}(s) &:= \int_0^\infty w_\pm(x)x^{s-1}\,\mathrm{d}x\\ &= \frac{1}{s\cdots(s+j-1)}\int_0^\infty w_\pm^{(j)}(x)x^{s+j-1}\,\mathrm{d}x \qquad (4.2)\\ &\ll_j \frac{Y}{X^{1-\sigma}}\bigg(\frac{X}{|s|Y}\bigg)^j \qquad \forall\ j\geqslant 1;\end{aligned}$$

- trivially $\widehat{w_\pm}(s) \ll X^\sigma$ and

$$\widehat{w_\pm}(1) = X + O(Y). \tag{4.3}$$

Obviously we have

$$\sum_n |\lambda_{\mathfrak f}(n)|^2 w_-(n) \leqslant \sum_{X<n\leqslant 2X} |\lambda_{\mathfrak f}(n)|^2 \leqslant \sum_n |\lambda_{\mathfrak f}(n)|^2 w_+(n). \tag{4.4}$$

Let the Dirichlet series associated with $|\lambda_{\mathfrak f}(n)|^2$ be defined as (see e.g. [13, (11)])

$$D(\mathfrak f\otimes\overline{\mathfrak f}, s) = \sum_{n=1}^\infty |\lambda_{\mathfrak f}(n)|^2 n^{-s}.$$

By the Mellin inversion formula

$$w_\pm(x) = \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \widehat{w_\pm}(s)x^{-s}\,\mathrm{d}s,$$

we write

$$\sum_n |\lambda_{\mathfrak f}(n)|^2 w_\pm(n) = \frac{1}{2\pi\mathrm{i}}\int_{(2)} \widehat{w_\pm}(s)D(\mathfrak f\otimes\overline{\mathfrak f}, s)\,\mathrm{d}s.$$

With the help of Cauchy's residue theorem, we obtain that

$$\sum_n \lambda_{\mathfrak f}(n)^2 w_\pm(n) = D_{\mathfrak f}\widehat{w_\pm}(1) + \frac{1}{2\pi\mathrm{i}}\int_{(\kappa)} \widehat{w_\pm}(s)D(\mathfrak f\otimes\overline{\mathfrak f}, s)\,\mathrm{d}s, \tag{4.5}$$

where $\frac12 < \kappa < 1$ and $D_{\mathfrak f} := \mathrm{Res}_{s=1}D(\mathfrak f\otimes\overline{\mathfrak f}, s)$. By (4.3), (4.2) with $j = 2$ and the convexity bound [13, Proposition 7]

$$D(\mathfrak f\otimes\overline{\mathfrak f}, s) \ll_{\mathfrak f,\varepsilon} (1+|\tau|)^{2\max(1-\sigma,0)+\varepsilon} \qquad (\tfrac12 < \sigma \leqslant 3),$$

we derive

$$\sum_n |\lambda_{\mathfrak f}(n)|^2 w_\pm(n) = D_{\mathfrak f}X + O_{\mathfrak f,\varepsilon}\big(Y + X^{1+\kappa}Y^{-1}\big).$$

Taking $\kappa = \frac12 + \varepsilon$ and $Y = X^{3/4}$, and combining the obtained estimation with (4.4), we find that

$$\sum_{X<n\leqslant 2X} |\lambda_{\mathfrak f}(n)|^2 = D_{\mathfrak f}X + O_{\mathfrak f,\varepsilon}\big(X^{3/4+\varepsilon}\big),$$

which implies (4.1) after a dyadic summation. □

Now we return to prove the theorem. Take $h = x^\eta$ where $\eta > \frac34$ is specified later. Lemma 4.1 gives

$$\text{(i)}\quad Ch \leqslant \sum_{x\leqslant n\leqslant x+h} \lambda_{\mathfrak f}(n)^2 \quad\text{ and }\quad \text{(ii)}\quad \sum_{x/m^2\leqslant t\leqslant (x+h)/m^2} \lambda_{\mathfrak f}(n)^2 \ll hm^{-3/2}$$

for any $m \leqslant \sqrt{x+h}$, where the positive constant C and the implied $\ll$-constant depend on $\mathfrak{f}$ and η only. Combining (i) with Lemma 3.1 leads to

$$Ch \leqslant \sum_{x\leqslant n\leqslant x+h} \lambda_{\mathfrak{f}}(n)^2 \leqslant C' \sum_{m\leqslant\sqrt{x+h}} \tau(m)^4 \sideset{}{^\flat}\sum_{x/m^2\leqslant t\leqslant(x+h)/m^2} \lambda_{\mathfrak{f}}(t)^2$$

where $\sum^\flat$ confines the running index over squarefree integers only and $C' > 0$ is a constant depending at most on $\mathfrak{f}$. By (ii) and the fact $\sum_{m\geqslant A} \tau(m)^4 m^{-3/2} \gg A^{-1/2+\varepsilon}$, we conclude that for a large enough constant A,

$$\sum_{m\leqslant A} \tau(m)^4 \sideset{}{^\flat}\sum_{x/m^2\leqslant t\leqslant(x+h)/m^2} \lambda_{\mathfrak{f}}(t)^2 \geqslant \{C/C' + O(A^{-1/2+\varepsilon})\}h \gg h$$

which is [13, (23)]. Thus, repeating the same argument (in [13, (24)-(26)]), we obtain [13, (26)] with a smaller admissible $h = x^\eta$ (here $\eta > \frac{3}{4}$ is required instead of $\eta > \frac{3}{4}+\varrho$).

Next we note that the new estimate (2.3) improves the upper bound $h^{3/4}x^\varepsilon$ in [13, (21) of Section 4.2] to $h^{1/2}x^\varepsilon$. Consequently, we get the new lower bound

$$x^{-1-\varrho}h^2 + O(h^{1/2}x^\varepsilon)$$

for [13, (27)]. The optimal choice of η is $\frac{2}{3}(1+\varrho)+\varepsilon$, and together with the constraint $\eta > \frac{3}{4}$, we choose

$$\eta = \max\left\{\tfrac{2}{3}(1+\varrho), \tfrac{3}{4}\right\} + \varepsilon.$$

We complete the proof of Theorem 1 with the same argument in remaining part of [13, Section 4.2].

5. Preparation for the truncated Voronoi formula

Applying the additive character to replace the congruence condition, that is,

$$Q^{-1}\sum_{d\mid Q}\sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \mathrm{e}\left(\frac{u(n-a)}{d}\right) = \delta_{n\equiv a\,(\mathrm{mod}\,Q)}$$

where $\delta_* = 1$ if $*$ holds and 0 otherwise, we have

$$\mathcal{S}_{\mathfrak{f}}^{\mathcal{A}}(x) := \sum_{\substack{n\leqslant x\\ n\equiv a(\mathrm{mod}\,Q)}} \lambda_{\mathfrak{f}}(n) = Q^{-1}\sum_{d\mid Q}\mathcal{S}_{\mathfrak{f}}(x, a/d), \tag{5.1}$$

where

$$\mathcal{S}_{\mathfrak{f}}(x, a/d) := \sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \mathrm{e}\left(\frac{-au}{d}\right)\sum_{n\leqslant x}\lambda_{\mathfrak{f}}(n)\mathrm{e}\left(\frac{nu}{d}\right). \tag{5.2}$$

Here $\sum^*_{u(\mathrm{mod}\,d)}$ denotes the sum over $u\,(\mathrm{mod}\,d)$ with $(u, d) = 1$. The inner sum over n is clearly associated with $L_{\mathfrak{f}}(s, u/d)$, thus we introduce the auxiliary function

$$\mathcal{L}_{\mathfrak{f}}(s, a/d) := \sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \mathrm{e}\Big(-\frac{au}{d}\Big)\, L_{\mathfrak{f}}(s, u/d). \tag{5.3}$$

The Dirichlet series associated to $\mathcal{S}^{\mathcal{A}}(x)$,

$$L_{\mathfrak{f}}(s, a, Q) := \sum_{\substack{n\geqslant 1\\ n\equiv a(\mathrm{mod}\,Q)}} \lambda_{\mathfrak{f}}(n)n^{-s} \tag{5.4}$$

is equal to

$$L_{\mathfrak{f}}(s,a,Q) = Q^{-1}\sum_{d\mid Q}\mathcal{L}_{\mathfrak{f}}(s,a/d). \tag{5.5}$$

Plainly $\mathcal{L}_{\mathfrak{f}}(s,a/d)$ satisfies a functional equation by (3.7),

$$q_d^s L_\infty(s)\mathcal{L}_{\mathfrak{f}}(s,a/d) = \mathrm{i}^{-(\ell+1/2)}q_d^{1-s}L_\infty(1-s)\widetilde{\mathcal{L}}_{\mathfrak{f}}(1-s,a/d) \tag{5.6}$$

where $\widetilde{L}_{\mathfrak{f}}(s,v/d)$ is defined as in (3.9) and

$$\widetilde{\mathcal{L}}_{\mathfrak{f}}(s,a/d) = \sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \mathrm{e}\Big(-\frac{au}{d}\Big)\,\widetilde{L}_{\mathfrak{f}}(s,\overline{u}/d) \qquad (u\overline{u}\equiv 1\,(\mathrm{mod}\,d)).$$

When $\Re e\, s>1$, we may express $\widetilde{\mathcal{L}}_{\mathfrak{f}}(s,a/d)$ as a Dirichlet series whose coefficients are products of $\lambda(n;d)$ and the Kloosterman-Salié sums. Indeed, by (3.9), we have

$$\widetilde{\mathcal{L}}_{\mathfrak{f}}(s,a/d) = \sum_{n\geqslant 1}\lambda(n;d)\mathrm{K}(a,n;d)n^{-s} \tag{5.7}$$

where (noting $v=\overline{u}\,(\mathrm{mod}\,d)$),

$$\mathrm{K}(a,n;d) := \sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \varpi_d(n,\overline{u})\mathrm{e}\Big(-\frac{au}{d}\Big). \tag{5.8}$$

By (3.10),

$$\mathrm{K}(a,n;d) = \begin{cases} \displaystyle\sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \varepsilon_u^{2\ell+1}\Big(\frac{d}{u}\Big)\,\mathrm{e}\Big(-\frac{a\overline{u}+nu}{4d}\Big) & \text{if } 4\mid d,\\ \displaystyle\sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \varepsilon_u^{2\ell+1}\Big(\frac{d}{u}\Big)\,\mathrm{e}\Big(-\frac{4a\overline{u}+nu}{4d}\Big) & \text{if } 2\parallel d,\\ \displaystyle\mathrm{i}^{\ell+1/2}\varepsilon_d^{-(2\ell+1)}\sideset{}{^*}\sum_{u\,(\mathrm{mod}\,d)} \Big(\frac{u}{d}\Big)\mathrm{e}\Big(-\frac{a\overline{u}+\overline{4}nu}{d}\Big) & \text{if } 2\nmid d.\end{cases}$$

Lemma 5.1. *Let $\tau(d)$ be the divisor function. We have*

$$|\mathrm{K}(a,n;d)| \ll (d,n)^{1/2}d^{1/2}\tau(d). \tag{5.9}$$

Moreover, for the case $2\nmid d$, if there exists $x\in\{a,n\}$ such that $(x,d)=1$, then

$$\mathrm{K}(a,n;d) = \mathrm{i}^{\ell+1/2}\varepsilon_d^{-2\ell}d^{1/2}\Big(\frac{x}{d}\Big)\sum_{y^2\equiv an(\mathrm{mod}\,d)}\mathrm{e}\Big(\frac{y}{d}\Big). \tag{5.10}$$

Proof. We express $\mathrm{K}(a,n;d)$ in terms of Kloosterman-Salié sums (see Appendix for their definitions), as follows:

$$\mathrm{K}(a,n;d) = \begin{cases} \overline{K_{2\ell+1}(n,a;d)} & \text{for } 4\mid d,\\ \frac14\overline{K_{2\ell+1}(n,a;4d)} & \text{for } 2\parallel d,\\ \mathrm{i}^{\ell+1/2}\varepsilon_d^{-(2\ell+1)}\,\overline{S(\overline{4}n,a;d)} & \text{for } 2\nmid d,\end{cases} \tag{5.11}$$

where in the case of $2\parallel d$, the range of summation is enlarged to a reduced residue system $(\mathrm{mod}\,4d)$. From (9.2) below, we have

$$|\mathrm{K}(a,n;d)| \ll (d,n)^{1/2}d^{1/2}\tau(d). \tag{5.12}$$

The formula (5.10) follows from the result in [9, Lemma 4.9] for the Salié sum. □

Lemma 5.2. *Let $d \geqslant 1$ and a be any integers. For any $\varepsilon > 0$, we have*

$$\mathcal{L}_{\mathfrak{f}}(\sigma + \mathrm{i}\tau, a/d) \ll d^{(3-\sigma)/2+2\varepsilon}(1+|\tau|)^{1-\sigma+2\varepsilon} \quad (-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon,\ \tau \in \mathbb{R}), \tag{5.13}$$

where the implied $\ll$-constant depends on $\mathfrak{f}$ and ε only.

Proof. Let $\Re e\, s = 1+\varepsilon$. By (3.3) and (3.6), we have trivially $L_{\mathfrak{f}}(s, u/d) \ll_\varepsilon 1$ and with (5.3), $\mathcal{L}_{\mathfrak{f}}(s, a/d) \ll_\varepsilon d$. Next for $\Re e\, s = -\varepsilon$, we infer from (5.6) and (5.7) that

$$\mathcal{L}_{\mathfrak{f}}(s, a/d) = \mathrm{i}^{-(\ell+1/2)} q_d^{1-2s} \frac{L_\infty(1-s)}{L_\infty(s)} \sum_{n\geqslant 1} \frac{\lambda(n;d)\mathrm{K}(a,n;d)}{n^{1-s}}.$$

Thus, with (5.12) and Stirling's formula, it follows that

$$\begin{aligned}\mathcal{L}_{\mathfrak{f}}(-\varepsilon + \mathrm{i}\tau, a/d) &\ll (d^{3/2}(1+|\tau|))^{1+\varepsilon} \sum_{n\geqslant 1} |\lambda(n;d)|(n,d)^{1/2} n^{-(1+\varepsilon)} \\ &\ll (d^{3/2}(1+|\tau|))^{1+\varepsilon}\end{aligned}$$

because $|\lambda(n;d)|(n,d)^{1/2} \leqslant |\lambda(n;d)|^2 + (n,d)$, implying that the last summation is

$$\ll \sum_{n\geqslant 1} |\lambda(n;d)|^2 n^{-(1+\varepsilon)} + \sum_{l|d} l^{-\varepsilon} \sum_{n\geqslant 1} n^{-(1+\varepsilon)} \ll \tau(d).$$

An application of Phragmén–Lindelöf principle completes the proof. □

6. Truncated Voronoi formula

This section is devoted to the Voronoi formulas. In order for a simpler form for the result, let us set, with the notation (5.8),

$$\phi_a(n,d) := \sqrt{q_d}\, \mathrm{i}^{-(\ell+1/2)} \mathrm{K}(a,n;d) \ll (n,d)^{1/2}\tau(d)d \tag{6.1}$$

by (5.12), and trivially $|\phi_a(n,d)| \leqslant \sqrt{2} d^{3/2}$. We have the following result.

Theorem 3. *Let $\ell \geqslant 2$ be an integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be an eigenform of all Hecke operators. Then for any $\varepsilon > 0$, we have*

$$\begin{aligned}\mathcal{S}_{\mathfrak{f}}(x, a/d) = {} & \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\leqslant M} \frac{\lambda(n;d)\phi_a(n,d)}{n^{3/4}} \cos\left(4\pi\frac{\sqrt{nx}}{q_d} - \frac{\ell+1}{2}\pi\right) \\ & + O_{\mathfrak{f},\varepsilon}\big(x^\varepsilon d^2 (x^{1/2+\varrho} M^{-1/2} + M^\varrho)\big)\end{aligned} \tag{6.2}$$

uniformly for $2 \leqslant M \leqslant x$ and $1 \leqslant d \leqslant x^{1/2}$, where ϱ is defined as in (3.5).

Moreover for $1 \leqslant Q \leqslant x^{1/2}$ and any integer a,

$$\begin{aligned}\mathcal{S}_{\mathfrak{f}}^{\mathcal{A}}(x) = {} & \frac{x^{1/4}}{\sqrt{2}\pi Q} \sum_{d|Q} \sum_{n\leqslant M} \frac{\lambda(n;d)\phi_a(n,d)}{n^{3/4}} \cos\left(4\pi\frac{\sqrt{nx}}{q_d} - \frac{\ell+1}{2}\pi\right) \\ & + O\big(x^\varepsilon Q (x^{1/2+\varrho} M^{-1/2} + M^\varrho)\big).\end{aligned}$$

In particular, for $Q \leqslant x^{\frac{1}{2}-\varrho}$ and any a,

$$\mathcal{S}_{\mathfrak{f}}^{\mathcal{A}}(x) \ll_{\mathfrak{f},\varepsilon} Q^{1/3} x^{(1+\varrho)/3+\varepsilon}. \tag{6.3}$$

Remark 3. It is shown in [15, Proposition 3.2] that $\mathcal{S}_{\mathfrak{f}}^{\mathbb{N}}(x) \ll x^{2/5+\varepsilon}$, which is superseded by the particular case $\mathcal{A} = \mathbb{N}$ (and $Q = 1$) of (6.3) for $\varrho = 1/6 + \varepsilon$ is admissible.

Proof. Let $d \leqslant x^{1/2}$, $1 \leqslant M \leqslant x$ and $T > 1$ be chosen as

$$T^2 = q_d^{-2} 4\pi^2 (M + 1/2)x \gg 1. \tag{6.4}$$

We apply the Perron formula (cf. [16, Corollary II.2.2.1]) to (5.3) with $\kappa := 1 + \varepsilon$, $\sigma_a = \alpha = 1$ and $B(n) = C_\varepsilon n^\varrho$ to write

$$\mathcal{S}_{\mathfrak{f}}(x, a/d) = \frac{1}{2\pi \mathrm{i}} \int_{\kappa - \mathrm{i}T}^{\kappa + \mathrm{i}T} \mathcal{L}_{\mathfrak{f}}(s, a/d) \frac{x^s}{s} \, \mathrm{d}s + O_{\mathfrak{f},\varepsilon}\bigg(\frac{dx^{1+\varrho}}{T}\bigg). \tag{6.5}$$

We deform the line of integration to the contour $\mathscr{L}$ joining the points $\kappa - \mathrm{i}T$, $-\varepsilon - \mathrm{i}T$, $-\varepsilon + \mathrm{i}T$, $\kappa + \mathrm{i}T$. Let $\mathscr{L}_{\mathrm{v}} := [-\varepsilon - \mathrm{i}T, -\varepsilon + \mathrm{i}T]$. By Lemma 5.2, the integrals over the horizontal segments of $\mathscr{L}$ are $\ll x^\varepsilon(xT^{-1} + d^{3/2})$, and the pole of the integrand at $s = 0$ gives $\mathcal{L}_{\mathfrak{f}}(0, a/d) \ll d^{3/2+\varepsilon}$. By the functional equation (5.6), the integral over $\mathscr{L}_{\mathrm{v}}$ equals

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}_{\mathrm{v}}} \mathcal{L}_{\mathfrak{f}}(s, a/d) \frac{x^s}{s} \, \mathrm{d}s = q_d \mathrm{i}^{-(\ell+1/2)} \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}_{\mathrm{v}}} \frac{L_\infty(1-s)}{L_\infty(s)} \widetilde{\mathcal{L}}_{\mathfrak{f}}(1-s, a/d) \bigg(\frac{\sqrt{x}}{q_d}\bigg)^{2s} \frac{\mathrm{d}s}{s}$$

By (5.7) and (6.1), we express (6.5) into

$$\mathcal{S}_{\mathfrak{f}}(x, a/d) = \frac{\sqrt{q_d}}{2\pi} \sum_{n \geqslant 1} \frac{\lambda(n; d)\phi_a(n, d)}{n} I_{\mathscr{L}_{\mathrm{v}}}\bigg(\frac{2\pi\sqrt{nx}}{q_d}\bigg) + O\bigg(\frac{dx^{1+\varrho}}{T} + d^{3/2}x^\varepsilon\bigg) \tag{6.6}$$

where

$$I_{\mathscr{L}_{\mathrm{v}}}(y) := \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}_{\mathrm{v}}} \frac{\Gamma(1 - s + \ell/2 - 1/4)}{\Gamma(s + \ell/2 - 1/4)} \cdot \frac{y^{2s}}{s} \, \mathrm{d}s.$$

Next we apply the stationary phase method to bound $I_{\mathscr{L}_{\mathrm{v}}}(y)$ for large y and give an asymptotic expansion in terms of trigonometric functions for small y.

With Stirling's formula, for $\tau > 0$, the integrand equals

$$\mathrm{e}^{\mathrm{i}\pi(\ell-1)/2} y^{2\sigma} \tau^{-2\sigma} \mathrm{e}^{2\mathrm{i}\tau \log(\mathrm{e}y/\tau)} \big\{1 + c_1\tau^{-1} + O\big(\tau^{-2}\big)\big\}$$

for any $|\tau| \geqslant 1$ and $|\sigma| \leqslant A$, where c_1 and $A > 0$ denote some suitable constants and the implied O-constant is independent of τ and y. Set $g(\tau) := 2\tau \log(\mathrm{e}y/\tau)$, then $g'(\tau) = 2\log(y/\tau)$. With the second mean value theorem for integrals (cf. [16, Theorem I.0.3]), we obtain for $y > T$ and $\sigma = -\varepsilon$,

$$\int_1^T y^{2\sigma} \tau^{-2\sigma} \mathrm{e}^{\mathrm{i}g(\tau)} \big\{1 + c_1\tau^{-1} + O\big(\tau^{-2}\big)\big\} \, \mathrm{d}\tau \ll T^{2\varepsilon} y^{2\sigma} \bigg|\log \frac{y}{T}\bigg|^{-1} + T^{2\varepsilon-1} y^{2\sigma}, \tag{6.7}$$

and for $y < T$ and $\sigma = \frac{1}{2} + \varepsilon$,

$$\int_T^\infty y^{2\sigma} \tau^{-2\sigma} \mathrm{e}^{\mathrm{i}g(\tau)} \big\{1 + c_1\tau^{-1} + O\big(\tau^{-2}\big)\big\} \, \mathrm{d}\tau \ll T^{-1-2\varepsilon} y^{2\sigma} \bigg|\log \frac{y}{T}\bigg|^{-1} + T^{-1-2\varepsilon} y^{2\sigma}. \tag{6.8}$$

For $n > M$, we infer by (6.7) that

$$I_{\mathscr{L}_{\mathrm{v}}}\bigg(\frac{2\pi\sqrt{nx}}{q_d}\bigg) \ll_k \bigg(\frac{x}{\sqrt{n}}\bigg)^{2\varepsilon} \bigg(\bigg|\log \frac{n}{M + 1/2}\bigg|^{-1} + d(Mx)^{-1/2}\bigg).$$

By $\lambda(n;d) \ll n^{\varrho+\varepsilon}$ from Lemma 3.1 and $|\phi_a(n,d)| \leqslant \sqrt{2}d^{3/2}$, it follows that

$$\sqrt{q_d}\sum_{n>M}\frac{|\lambda(n;d)\phi_a(n,d)|}{n^{1+\varepsilon}}\left|\log\frac{n}{M+1/2}\right|^{-1} \ll d^2M^{\varrho}\sum_{M<n<2M}|n-(M+1/2)|^{-1} \ll d^2M^{\varrho+\varepsilon}.$$

Consequently we deduce that

$$\frac{\sqrt{q_d}}{2\pi}\sum_{n>M}\frac{\lambda_{\mathfrak{h}}(n)\phi_a(n,d)}{n}I_{\mathscr{L}_{\mathrm{v}}}\left(\frac{2\pi\sqrt{nx}}{q_d}\right) \ll x^{\varepsilon}d^2M^{\varrho}+x^{\varepsilon}d^2(Mx)^{-1/2}. \tag{6.9}$$

For $n \leqslant M$, we complete the path $\mathscr{L}_{\mathrm{v}}$ to the contour $\mathscr{L}_{\mathrm{v}}^*$ so as to apply [1, Lemma 1], where $\mathscr{L}_{\mathrm{v}}^*$ is the positively oriented contour consisting of $\mathscr{L}_{\mathrm{v}}$, $\mathscr{L}_{\mathrm{v}}^{\pm}$ and $\mathscr{L}_{\mathrm{h}}^{\pm}$ with

$$\mathscr{L}_{\mathrm{v}}^{\pm} := [\tfrac12+\varepsilon\pm\mathrm{i}T,\, \tfrac12+\varepsilon\pm\mathrm{i}\infty), \qquad \mathscr{L}_{\mathrm{h}}^{\pm} := [-\varepsilon\pm\mathrm{i}T,\, \tfrac12+\varepsilon\pm\mathrm{i}T].$$

Correspondingly we denote by $I_{\mathscr{L}_{\mathrm{v}}^{\pm}}$ and $I_{\mathscr{L}_{\mathrm{h}}^{\pm}}$ the integrals over these segments. By (6.8), the integral over the vertical line segments $\mathscr{L}_{\mathrm{v}}^{\pm}$ is

$$I_{\mathscr{L}_{\mathrm{v}}^{\pm}} \ll x^{\varepsilon}\left(\frac{n}{M}\right)^{1/2}\left|\log\frac{n}{M+1/2}\right|^{-1},$$

while for the horizontal segments, $I_{\mathscr{L}_{\mathrm{h}}^{\pm}}$ contributes at most $O((n/M)^{\varepsilon})$. Thus

$$\begin{aligned}&\frac{\sqrt{q_d}}{2\pi}\sum_{n\leqslant M}\frac{\lambda(n;d)\phi_a(n,d)}{n}\left(I_{\mathscr{L}_{\mathrm{v}}^{\pm}}+I_{\mathscr{L}_{\mathrm{h}}^{\pm}}\right)\\ &\ll x^{\varepsilon}d^2M^{\rho-1/2}\sum_{M/2\leqslant n\leqslant M}n^{-1/2}\left|\log\frac{M+1/2}{M+1/2-n}\right|^{-1}\\ &\ll x^{\varepsilon}d^2M^{\varrho}.\end{aligned} \tag{6.10}$$

Inserting (6.10) and (6.9) into (6.6), we get from our choice of T,

$$\begin{aligned}\mathcal{S}_{\mathfrak{f}}(x,a/d) = {}&\frac{\sqrt{q_d}}{2\pi}\sum_{1\leqslant n\leqslant M}\frac{\lambda(n;d)\phi_a(n,d)}{n}I_{\mathscr{L}_{\mathrm{v}}^*}\left(\frac{2\pi\sqrt{nx}}{q_d}\right)\\ &+O\big(x^{\varepsilon}d^2(x^{1/2+\varrho}M^{-1/2}+M^{\rho})\big).\end{aligned} \tag{6.11}$$

Now all the poles of the integrand in

$$I_{\mathscr{L}_{\mathrm{v}}^*}(y) := \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{L}_{\mathrm{v}}^*}\frac{\Gamma(1-s+\ell/2-1/4)\Gamma(s)}{\Gamma(s+\ell/2-1/4)\Gamma(s+1)}y^{2s}\,\mathrm{d}s.$$

lie on the right of the contour $\mathscr{L}_{\mathrm{v}}^*$. After a change of variable s into $1-s$, we have

$$I_{\mathscr{L}_{\mathrm{v}}^*}(y) = \frac{1}{\pi}I_0\big(y^2\big),$$

with

$$I_0(y) := \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{L}_{\varepsilon}}\frac{\Gamma(s+(2\ell-1)/4)\Gamma(1-s)}{\Gamma(1-s+(2\ell-1)/4)\Gamma(2-s)}y^{1-s}\,\mathrm{d}s.$$

Here $\mathscr{L}_{\varepsilon}$ consists of the line $s=\frac12-\varepsilon+\mathrm{i}\tau$ with $|\tau|\geqslant T$, together with three sides of the rectangle whose vertices are $\frac12-\varepsilon-\mathrm{i}T$, $1+\varepsilon-\mathrm{i}T$, $1+\varepsilon-\mathrm{i}T$ and $\frac12-\varepsilon+\mathrm{i}T$. Clearly our I_0 is a particular case of I_{ρ} defined in [1, Lemma 1], corresponding to the choice of parameters $A=\delta=N=\omega=\alpha_1=1$, $\beta_1=\mu=(\ell-2)/4$, $\rho=m=0$, $a=-\frac{3}{4}$, $c_0=\frac{1}{2}$, $h=2$, $k_0=-(\ell+1)/2$. It hence follows that

$$I_{\mathscr{L}_{\mathrm{v}}^*}\bigg(\frac{2\pi\sqrt{nx}}{q_d}\bigg)=e_0'\sqrt{\frac{2\pi}{q_d}}(nx)^{1/4}\cos\bigg(4\pi\frac{\sqrt{nx}}{q_d}-\frac{\ell+1}{2}\pi\bigg)+O\big(d^{1/2}(nx)^{-1/4}\big). \tag{6.12}$$

The value of e_0' [1, Lemma 1] is $1/\sqrt{\pi}$, and the main term in (6.2) follows from (6.12) and (6.11). With a simple checking, the O-term in (6.12) gives a term that will be absorbed in (6.11).

Finally we set $M=Q^{4/3}x^{(1+4\rho)/3}$ and note from (6.1) that

$$\sum_{n\leqslant M}\frac{|\lambda(n;d)\phi_a(n,d)|}{n^{3/4}}\ll d^{1+\varepsilon}\sum_{n\leqslant M}|\lambda(n;d)|^2n^{-3/4}+d^{1+\varepsilon}\sum_{n\leqslant M}(n,d)n^{-3/4},$$

which is $\ll x^{\varepsilon}dM^{1/4}$ with (3.3). □

7. Preparation for the proof of Theorem 2

We consider odd Q only, then $q_d=2d$ and $\lambda(n;d)=\lambda_{\mathfrak{h}}(n)$ for all $d\mid Q$. The idea of proof is the same as in Heath-Brown & Tsang [5], however, some new technicality arises because of the new frequencies ($\sqrt{n}/q_d$ rather than $\sqrt{n}$). Consequently, instead of $\sqrt{1}$, we shall apply their argument to the frequency $\sqrt{n_0}/Q$ where $n_0=2^jf_0$ with $j\geqslant 0$ and f_0 squarefree, and simultaneously, require the coefficient $\lambda_{\mathfrak{h}}(n_0)\phi_a(n_0,Q)$ to be non-vanishing. We can guarantee the existence of n_0 under certain circumstances.

For convenience, let us recall our notation (specialized to this case $2\nmid d$):

$$\mathcal{S}_{\mathfrak{f}}^{A}(x)=\sum_{\substack{n\leqslant x\\ n\equiv a(\mathrm{mod}\,Q)}}\lambda_{\mathfrak{f}}(n)\quad\text{and}\quad\mathcal{S}_{\mathfrak{f}}(x,a/d):=\sum_{n\leqslant x}\lambda_{\mathfrak{f}}(n)R_d(n-a).$$

where $R_d(m)=\sum^*_{u(\mathrm{mod}\,d)}\mathrm{e}\,(mu/d)$ is the Ramanujan sum. Their associated Dirichlet series are

$$L_{\mathfrak{f}}(s,a,Q):=\sum_{\substack{n\geqslant 1\\ n\equiv a(\mathrm{mod}\,Q)}}\lambda_{\mathfrak{f}}(n)n^{-s}\quad\text{and}\quad\mathcal{L}_{\mathfrak{f}}(s,a/d):=\sum_{n\geqslant 1}\lambda_{\mathfrak{f}}(n)R_d(n-a)n^{-s}.$$

Moreover, $L_{\mathfrak{f}}(s,a,Q)=Q^{-1}\sum_{d\mid Q}\mathcal{L}_{\mathfrak{f}}(s,a/d)$ and

$$(2d)^sL_\infty(s)\mathcal{L}_{\mathfrak{f}}(s,a/d)=\mathrm{i}^{-(\ell+1/2)}(2d)^{1-s}L_\infty(1-s)\widetilde{\mathcal{L}}_{\mathfrak{f}}(1-s,a/d)$$

where

$$\widetilde{\mathcal{L}}_{\mathfrak{f}}(s,a/d):=\sum_{n\geqslant 1}\lambda_{\mathfrak{h}}(n)\mathrm{K}(a,n;d)n^{-s}.$$

Lemma 7.1. *Under the assumption that $\{\lambda_{\mathfrak{f}}(n)\}_{n\in\mathbb{N}}$ is a real sequence, for all a,d, the sequences $\{\mathrm{i}^{-(\ell+1/2)}\lambda_{\mathfrak{h}}(n)\mathrm{K}(a,n;d)\}_{n\in\mathbb{N}}$ are real.*

Proof. Since the Ramanujan sum $R_d(m)$ is real-valued, $\mathcal{L}_{\mathfrak{f}}(s,a/d)$ is real-valued for $s\in(1,\infty)$ under the given assumption. The holomorphicity of $\mathcal{L}_{\mathfrak{f}}(s,a/d)$ implies that $\overline{\mathcal{L}_{\mathfrak{f}}(\overline{s},a/d)}$ is holomorphic. Thus $\overline{\mathcal{L}_{\mathfrak{f}}(\overline{s},a/d)}=\mathcal{L}_{\mathfrak{f}}(s,a/d)$ on $\mathbb{C}$ (as they are equal on $(1,\infty)$). The lemma follows. □

Lemma 7.2. *When the sequence* $\{\lambda_{\mathfrak{f}}(n)\}_{n\in\mathcal{A}}$ *contains nonzero terms, the function* $\mathcal{L}_{\mathfrak{f}}(s,a/d)$ *is non-identically zero for all* $d \mid Q$.

Proof. Suppose not, say, $\mathcal{L}_{\mathfrak{f}}(s,a/d_0)\equiv 0$. Then

$$\sum_{\substack{n\geqslant 1\\ n\equiv a(\operatorname{mod} Q)}} \lambda_{\mathfrak{f}}(n)n^{-s} = Q^{-1}\sum_{\substack{d|Q\\ d\neq d_0}} \mathcal{L}_{\mathfrak{f}}(s,a/d) = \sum_{n\geqslant 1} n^{-s}\lambda_{\mathfrak{f}}(n)Q^{-1}\sum_{\substack{d|Q\\ d\neq d_0}} R_d(n-a).$$

With the standard formula for the Ramanujan sum, we infer that

$$\delta_{n\equiv a(\operatorname{mod} Q)}\lambda_{\mathfrak{f}}(n) = \lambda_{\mathfrak{f}}(n)Q^{-1}\sum_{\substack{d|Q\\ d\neq d_0}}\sum_{\substack{\delta|d\\ (d/\delta)|(n-a)}} \mu(\delta)(d/\delta) \qquad \forall\; n\geqslant 1.$$

Take $n\equiv a\,(\operatorname{mod} Q)$ such that $\lambda_{\mathfrak{f}}(n)\neq 0$. We obtain that

$$Q-\phi(d_0) = \sum_{\substack{d|Q\\ d\neq d_0}} \phi(d) = \sum_{\substack{d|Q\\ d\neq d_0}}\sum_{\delta|d}\mu(\delta)(d/\delta) = Q.$$

Contradiction arises. □

Proposition 1. *Let* $Q\geqslant 1$ *be odd and* $0\leqslant a<d$. *Suppose* $n_0=2^j f_0$ *with* f_0 *squarefree and* $j\geqslant 0$ *is an integer such that*

$$\lambda_{\mathfrak{h}}(n_0)\phi_a(n_0,Q)\neq 0. \tag{7.1}$$

Then there are constants $c_0=c_0(\mathfrak{f},Q,n_0)$ *and* $x_0=x_0(\mathfrak{f},Q,n_0)$ *such that* $\mathcal{S}_{\mathfrak{f}}^{\mathcal{A}}(x)$ *attains at least one sign change in the interval* $[x,x+c_0\sqrt{x}]$ *for all* $x\geqslant x_0$.

Proof. Let α a parameter determined later and T be any sufficiently large number. Set

$$F_{\mathfrak{f}}(t+\alpha u) := \pi\sqrt{Q}\frac{S_{\mathfrak{f}}^{\mathcal{A}}((Q(t+\alpha u))^2)}{\sqrt{t+\alpha u}} \qquad (t\in[T,2T],\, u\in[-1,1]).$$

By Theorem 3 with $M=(QT)^2$, we deduce that

$$\begin{aligned}F_{\mathfrak{f}}(t+\alpha u) = &\sum_{d|Q}\sum_{n\leqslant (QT)^2}\frac{\lambda_{\mathfrak{h}}(n)\phi_a(n,d)}{n^{3/4}}\cos\left(\pi(t+\alpha u)\frac{Q\sqrt{n}}{d}-\frac{\ell+1}{2}\pi\right)\\ &+O\big(Q(QT)^{2\varrho-1/2+\varepsilon}\big).\end{aligned}$$

Let $\tau=1$ or -1, and define

$$k_\tau(u) := (1-|u|)(1+\tau\cos(2\pi\alpha\sqrt{n_0}u)).$$

Then as in the proof of [12, Lemma 3.2], for any $n\in\mathbb{N}$ and $t\in\mathbb{R}$, the integral

$$r_n = r_n(\alpha,\tau,t) := \int_{-1}^{1} k_\tau(u)\cos\left(2\pi(t+\alpha u)\frac{Q\sqrt{n}}{d}-\frac{\ell+1}{2}\pi\right)\mathrm{d}u$$

satisfies

$$\begin{aligned} r_n = &\,\delta_{Q\sqrt{n}=d\sqrt{n_0}}\cdot\frac{\tau}{2}\cos\left(2\pi t\sqrt{n_0}-\frac{\ell+1}{2}\pi\right)\\ &+O\left(\min\left(1,\frac{1}{\alpha^2 n}\right)+\delta_{Q\sqrt{n}\neq d\sqrt{n_0}}\min\left(1,\frac{1}{(\alpha_{n,d}^-)^2}\right)\right),\end{aligned} \tag{7.2}$$

where $\alpha_{n,d}^- = \alpha|Q\sqrt{n}-d\sqrt{n_0}|/d$, $\delta_*=1$ if $*$ holds, or 0 otherwise. The O-constant is absolute.

Observe that $Q\sqrt{n} = d\sqrt{n_0}$ if and only if $2^j f_0 = (Q/d)^2 n$ which is equivalent to $n = 2^j f_0 = n_0$ and $d = Q$ since f_0 is squarefree and Q/d is odd. Following from (7.2) and (7.2), the integral

$$J_\tau(t) = \int_{-1}^{1} F_{\mathfrak{f}}(t + \alpha u) k_\tau(u)\,\mathrm{d}u$$

can be written as

$$J_\tau(t) = \frac{\tau}{2} \frac{\lambda_{\mathfrak{h}}(n_0)\phi_a(n_0, Q)}{n_0^{3/4}} \cos\left(2\pi t\sqrt{n_0} - \frac{\ell+1}{2}\pi\right) + \mathrm{E} + O\big(Q(QT)^{2\varrho-1/2+\varepsilon}\big) \tag{7.3}$$

where

$$\mathrm{E} \ll \frac{1}{\alpha^2} \sum_{d|Q} \sum_{n\leqslant (QT)^2} \frac{|\lambda_{\mathfrak{h}}(n)\phi_a(n,d)|}{n^{7/4}} + \sum_{d|Q} \frac{d^2}{\alpha^2} \sum_{\substack{n\leqslant (QT)^2 \\ Q\sqrt{n}\neq d\sqrt{n_0}}} \frac{|\lambda_{\mathfrak{h}}(n)\phi_a(n,d)|}{n^{3/4}|Q\sqrt{n} - d\sqrt{n_0}|^2}.$$

Using the bounds $\phi_a(n,d) \ll d^{3/2}$ and $\lambda_{\mathfrak{h}}(n) \ll n^{\varrho}$, a little calculation gives

$$\mathrm{E} \ll Q^3 n_0^{\varrho+1/4} \alpha^{-2}.$$

Let $A_0 := |\lambda_{\mathfrak{h}}(n_0)\phi_a(n_0, Q)| n_0^{-3/4}$, which is > 0. Fix a sufficiently large $\alpha = \alpha(\mathfrak{f}, n_0, Q)$, so that E is $< \frac{1}{8}A_0$, and then a sufficiently large $T_0 = T_0(\mathfrak{f}, n_0, Q, \alpha)$ such that the O-term $O\big(Q(QT)^{2\varrho-1/2+\varepsilon}\big)$ is $\leqslant \frac{1}{8}A_0$ for all $T \geqslant T_0$. Now observe that for any $m \in \mathbb{N}$, the absolute value of the cosine factor is $1/\sqrt{2}$ if $t = t_m$ where

$$t_m := (m + \tfrac{1}{8}) n_0^{-1/2}.$$

This implies $|J_\tau(t_m)| > \frac{1}{4}(\sqrt{2} - 1)A_0 > 0$ whenever $t_m > T_0 + \alpha$. Since $J_\pm(t_m)$ are of opposite signs and the kernel function k_τ is nonnegative, there is a pair of $t_m^\pm \in [t_m - \alpha, t_m + \alpha]$ for which $\pm F_{\mathfrak{f}}(t_m^\pm) > 0$. Equivalently, $\mathcal{S}_{\mathfrak{f}}^{\mathcal{A}}(y)$ attains a sign change in every interval of the form $[(Q(t_m - \alpha))^2, (Q(t_m + \alpha))^2]$ whose length is $\ll \alpha(Q^2 t_m) \ll_{\mathfrak{f},Q,n_0} \sqrt{x}$ when $x = (Qt_m)^2$. Our result follows readily. □

8. Proof of Theorem 2

In view of Proposition 1, the main task is to study the condition $\lambda_{\mathfrak{h}}(n_0)\phi_a(n_0, Q)$. Recall $\phi_a(n, Q) = \sqrt{2Q}\,\mathrm{i}^{-(\ell+1/2)}\mathrm{K}(a, n; Q)$ by (6.1). Clearly, $\phi_a(n, 1) = \sqrt{2}$. In general, we have by Lemma 9.1 (2),

$$\phi_a(n, Q) = \sqrt{2Q}\,\varepsilon_Q^{-(2\ell+1)} \prod_{p^\alpha \| Q} S(n\overline{4Q_p}, a\overline{Q_p}; p^\alpha) \tag{8.1}$$

where $S(m, n; c)$ is defined as in (9.1), $Q_p = Q/p^\alpha$ and $\overline{x}x \equiv 1 \,(\mathrm{mod}\, p^\alpha)$ for each term inside the product, $\forall\ p^\alpha \parallel Q$.

♠ Case 1. $Q = 1$. It suffices to find a squarefree t and a $j \geqslant 0$ such that $\lambda_{\mathfrak{h}}(2^j t) \neq 0$. By Lemma 7.2, $\mathcal{L}_{\mathfrak{f}}(s, 1)$ and thus $\widetilde{\mathcal{L}}_{\mathfrak{f}}(s, 1) = \sum_{n\geqslant 1} \lambda_{\mathfrak{h}}(n) n^{-s}$ are not identical to the zero function. Thus $\lambda_{\mathfrak{h}}(n) \neq 0$ for some $n \in \mathbb{N}$. Write $n = 2^j t m^2$ where t is squarefree and m is odd, $\lambda_{\mathfrak{h}}(2^j t) \neq 0$ from (3.4).

♠ Case 2. $a = 0$ and $p^\alpha \,\|\, Q$ implies α being odd. By Lemma 9.1 (2)-(3) and (8.1), $\phi_0(n,Q) = 0$ if $(n,Q) > 1$. Repeating the argument in Case 1, we get $\lambda_{\mathfrak{h}}(n)\phi_0(n,Q) \neq 0$ for some $n \in \mathbb{N}$. This n has to be coprime with Q. Write $n = 2^j tm^2$ with squarefree t and odd m, then $\lambda_{\mathfrak{h}}(2^j t) \neq 0$ (from $\lambda_{\mathfrak{h}}(2^j tm^2) \neq 0$) and $\phi_0(2^j t, Q) \neq 0$ because

$$S(hk,0;Q) = \left(\frac{h}{Q}\right) S(k,0;Q)$$

if $(h,Q) = 1$, from the definition of the Salié sum.

♠ Case 3. $(a,Q) = 1$ and $p^2 \mid Q$, $\forall\, p|Q$. The argument is similar to the previous cases – firstly finding $n = 2^j tm^2$, with squarefree t and odd m, for which $\lambda_{\mathfrak{h}}(n)\phi_0(n,Q) \neq 0$. But now we need (5.10) to analyze the Salié sum, which gives

$$\phi_a(2^j tm^2, Q) = \sqrt{2} Q \varepsilon_Q^{-2\ell} \left(\frac{a}{Q}\right) c_{a2^j t}(m,Q)$$

where

$$c_b(m,d) = \sum_{\substack{y \,(\mathrm{mod}\, d) \\ y^2 \equiv bm^2 \,(\mathrm{mod}\, d)}} \mathrm{e}\left(\frac{y}{d}\right). \tag{8.2}$$

As in (8.1), we have the factorization

$$c_{a2^j t}(m,Q) = \prod_{p^\alpha \| Q} c_{\overline{Q_p} a 2^j t}(m, p^\alpha)$$

and the lemma below assures $(m,Q) = 1$ and $\phi_a(2^j t, Q) \neq 0$ when $\phi_a(2^j tm^2, Q) \neq 0$. Hence this case is also complete.

Lemma 8.1. *Let $b \in \mathbb{Z}$, p an odd prime and $\alpha \geqslant 2$. Define $c_b(m,p^\alpha)$ as in* (8.2). *Then*

(i) $c_b(m,p^\alpha) = 0$ *if $p \mid m$, and*
(ii) $c_b(1,p^\alpha) \neq 0$ *if $c_b(m,p^\alpha) \neq 0$ with $p \nmid m$.*

Proof. (i) Write $m = p^\beta m'$ where $p \nmid m'$.

- $\alpha = 2\gamma \leqslant 2\beta$. Then

$$c_b(m,p^\alpha) = \sum_{y^2 \equiv 0 (\mathrm{mod}\, p^\alpha)} \mathrm{e}\left(\frac{y}{p^\alpha}\right) = \sum_{l (\mathrm{mod}\, p^\gamma)} \mathrm{e}\left(\frac{l}{p^\gamma}\right) = 0.$$

- $\alpha = 2\gamma + 1 \leqslant 2\beta$. Then y is of the form $y = lp^{\gamma+1}$, and as $\gamma \geqslant 1$,

$$c_b(m,p^\alpha) = \sum_{y^2 \equiv 0 (\mathrm{mod}\, p^\alpha)} \mathrm{e}\left(\frac{y}{p^\alpha}\right) = \sum_{l (\mathrm{mod}\, p^\gamma)} \mathrm{e}\left(\frac{l}{p^\gamma}\right) = 0.$$

- $\alpha > 2\beta \geqslant 2$. Then $y = lp^\beta$ and thus

$$\begin{aligned} c_b(m,p^\alpha) &= \sum_{l^2\equiv bm'^2 (\operatorname{mod} p^{\alpha-2\beta})} \sum_{y\equiv p^\beta l (\operatorname{mod} p^\alpha)} \mathrm{e}\Big(\frac{y}{p^\alpha}\Big) \\ &= \sum_{l^2\equiv bm'^2 (\operatorname{mod} p^{\alpha-2\beta})} \sum_{t (\operatorname{mod} p^\beta)} \mathrm{e}\Big(\frac{l+tp^{\alpha-2\beta}}{p^{\alpha-\beta}}\Big) \\ &= \sum_{l^2\equiv bm'^2 (\operatorname{mod} p^{\alpha-2\beta})} \mathrm{e}\Big(\frac{l}{p^{\alpha-\beta}}\Big) \sum_{t (\operatorname{mod} p^\beta)} \mathrm{e}\Big(\frac{t}{p^\beta}\Big) \\ &= 0. \end{aligned}$$

(ii) Suppose $c_b(m,p^\alpha) \neq 0$ where $(m,p)=1$. We may assume $p^2 \nmid b$, for otherwise, $c_b(m,p^\alpha) = c_{b/p^2}(mp,p^\alpha) = 0$ by (i). Also $p \,\|\, b$ cannot happen because, when $\alpha \geqslant 2$, $p^2 \mid b$ if $p \mid b$ and $y^2 \equiv bm^2 (\operatorname{mod} p^\alpha)$ has solutions. Thus $p \nmid b$.

Now $c_b(m,p^\alpha) \neq 0$ implies the congruence $y^2 \equiv bm^2 \,(\operatorname{mod} p^\alpha)$ is soluble, and with $(m,p)=1$, $y^2 \equiv b \,(\operatorname{mod} p^\alpha)$ has two solutions, say, $\pm y_0$ and $p \nmid y_0$. We see that

$$\sum_{y^2\equiv b (\operatorname{mod} p^\alpha)} \mathrm{e}\Big(\frac{y}{p^\alpha}\Big) = 2\cos\Big(2\pi \frac{y_0}{p^\alpha}\Big) \neq 0$$

because otherwise, $y_0/p^\alpha = (2r+1)/4$ for some $r \in \mathbb{Z}$ or equivalently, $4y_0 = (2r+1)p^\alpha$ which contradicts to $p \nmid y_0$. $\square$

9. Appendix

Let us denote, as in [8, Section 3], the Kloosterman-Salié sum by

$$K_{2\ell+1}(m,n;c) := \sum_{d \,(\operatorname{mod} c)} \varepsilon_d^{-(2\ell+1)} \Big(\frac{c}{d}\Big) \mathrm{e}\Big(\frac{md + n\overline{d}}{c}\Big)$$

and

$$S(m,n;c) := \sum_{x \,(\operatorname{mod} c)} \Big(\frac{x}{c}\Big) \mathrm{e}\Big(\frac{mx + n\overline{x}}{c}\Big), \tag{9.1}$$

where $c \in \mathbb{N}$ and $m,n \in \mathbb{Z}$. Then we have the following estimate,

$$|K_{2\ell+1}(n,m;d)| \qquad \text{and} \qquad |S(m,n;d)| \leqslant d^{1/2}\tau(d)(d,n,m)^{1/2} \tag{9.2}$$

where $\tau(n)$ is the divisor function. This follows from the well-known Weil's bound for Kloosterman sums and the following lemma.

Lemma 9.1. *We have the following results:*

(a) *Let $c = qr$ with $r \equiv 0 \,(\operatorname{mod} 4)$ and $(q,r)=1$. Then*

$$K_{2\ell+1}(m,n;c) = K_{2\ell+2-q}(m\overline{q}, n\overline{q}; r) S(m\overline{r}, n\overline{r}; q)$$

where $q\overline{q} \equiv 1 \,(\operatorname{mod} r)$ and $r\overline{r} \equiv 1 \,(\operatorname{mod} q)$.

(b) *Let q be odd, $q = uv$ with $(u,v)=1$. Then*

$$S(m,n;q) = S(m\overline{u}, n\overline{u}; v) S(m\overline{v}, n\overline{v}; u)$$

where $u\overline{u} \equiv 1 \,(\operatorname{mod} v)$ and $v\overline{v} \equiv 1 \,(\operatorname{mod} u)$.

(c) *For an odd prime p and odd α, if $p \mid m$, then $S(m,0;p^\alpha)=0$.*
(d) *If $(c,2)=1$, then $|S(m,n;c)| \leqslant (m,n,c)^{1/2}c^{1/2}\tau(c)$.*
(e) *Let $4|r|2^\infty$. Then $|K_{2\ell+1}(m,n;r)| \leqslant (m,n,r)^{1/2}r^{1/2}\tau(r)$.*

Proof. (a) See [8, p. 390, Lemma 2].
(b) See [8, p. 390, Lemma 3].
(c) By definition, for odd α, we have

$$S(m,0;p^\alpha)=\sum_{x\,(\mathrm{mod}\,p^\alpha)}\left(\frac{x}{p}\right)\mathrm{e}\left(\frac{mx}{p^\alpha}\right).$$

When $\alpha=1$, $S(m,0;p^\alpha)=\sum_{x\,(\mathrm{mod}\,p^\alpha)}\left(\frac{x}{p}\right)=0$ as $p\mid m$. Suppose $\alpha\geqslant 3$. Putting $x=lp+v$, we get

$$\sum_{l\,(\mathrm{mod}\,p^{\alpha-1})}\mathrm{e}\left(\frac{ml}{p^{\alpha-1}}\right)\sum_{v\,(\mathrm{mod}\,p)}\left(\frac{v}{p}\right)\mathrm{e}\left(\frac{mv}{p}\right)=0.$$

(d) Iwaniec [9, Section 4.6] handled the case $(c,2n)=1$, and thus $(c,2m)=1$ too by symmetry. Together with (b), it suffice to deal with $p\mid(m,n)$ and c is a power of p.

Consider $S:=S(p^am,p^{a+b}n;p^{a+t})$ where $b\geqslant 0$, $p\nmid mn$, $a,t\geqslant 1$ and $a+t$ is odd. (The case that $a+t$ is even is done with the classical Kloosterman sum.) Clearly,

$$S=\sum_{d\,(\mathrm{mod}\,p^{a+t})}\left(\frac{d}{p}\right)\mathrm{e}\left(\frac{md+p^bn\overline{d}}{p^t}\right)=\left(\frac{m}{p}\right)\sum_{d\,(\mathrm{mod}\,p^{a+t})}\left(\frac{d}{p}\right)\mathrm{e}\left(\frac{d+p^bmn\overline{d}}{p^t}\right).$$

Mimicking Iwaniec's proof in [8, p. 67] (in fact attributed to Sarnak), we consider

$$F(x)=\sum_{d\,(\mathrm{mod}\,p^{a+t})}\left(\frac{d}{p}\right)\mathrm{e}\left(\frac{x^2d+p^bmn\overline{d}}{p^t}\right).$$

and its Fourier transform

$$\widehat{F}(y)=\sum_{x\,(\mathrm{mod}\,p^t)}F(x)\mathrm{e}\left(-\frac{xy}{p^t}\right).$$

As in [8, p. 67], we obtain $\widehat{F}(y)=g(1,p^t)G_t(4mnp^b-y^2)$ where

$$G_t(4mnp^b-y^2)=\sum_{d\,(\mathrm{mod}\,p^{a+t})}\left(\frac{d}{p}\right)^{t+1}\mathrm{e}\left(\frac{d(4mnp^b-y^2)}{p^t}\right).$$

<u>Case 1: t is odd.</u> Then

$$\begin{aligned}G_t(4mnp^b-y^2)&=\sum_{d\,(\mathrm{mod}\,p^{a+t})}^{*}\mathrm{e}\left(\frac{d(4mnp^b-y^2)}{p^t}\right)\\&=\sum_{r=0,1}(-1)^rp^a\sum_{d\,(\mathrm{mod}\,p^{t-r})}\mathrm{e}\left(\frac{d(4mnp^b-y^2)}{p^{t-r}}\right).\end{aligned}$$

Since

$$\sum_{d\,(\mathrm{mod}\,p^{t-r})}\mathrm{e}\left(\frac{d(4mnp^b-y^2)}{p^{t-r}}\right)=p^{t-r}\delta_{y^2\equiv 4mnp^b\,(\mathrm{mod}\,p^{t-r})},$$

we conclude

$$\widehat{F}(y) = g(1, p^t) \sum_{r=0,1} (-1)^r p^{a+t-r} \delta_{y^2 \equiv 4mnp^b \,(\mathrm{mod}\, p^{t-r})}$$

and

$$\begin{aligned} F(x) &= p^{-t} \sum_{y \,(\mathrm{mod}\, p^t)} \widehat{F}(y) \mathrm{e}\Big(\frac{xy}{p^t}\Big) \\ &= g(1, p^t) \sum_{r=0,1} (-1)^r p^{a-r} \sum_{\substack{y \,(\mathrm{mod}\, p^t) \\ y^2 \equiv 4mnp^b \,(\mathrm{mod}\, p^{t-r})}} \mathrm{e}\Big(\frac{xy}{p^t}\Big). \end{aligned}$$

As $|g(1, p^t)| \leqslant p^{t/2}$ by [9, (4.43)], we see that $|F(1)| \leqslant 2p^{a+t/2}$.

Case 2: t is even. Then

$$\begin{aligned} G_t(4mnp^b - y^2) &= \sum_{d \,(\mathrm{mod}\, p^{a+t})} \Big(\frac{d}{p}\Big) \mathrm{e}\Big(\frac{d(4mnp^b - y^2)}{p^t}\Big) \\ &= \sum_{u \,(\mathrm{mod}\, p^{a+t-1})} \mathrm{e}\Big(\frac{u(4mnp^b - y^2)}{p^{t-1}}\Big) \sum_{v \,(\mathrm{mod}\, p)} \Big(\frac{v}{p}\Big) \mathrm{e}\Big(\frac{v(4mnp^b - y^2)}{p^{t-1}}\Big). \end{aligned}$$

The first sum does not vanish only when $y^2 \equiv 4mn \,(\mathrm{mod}\, p^{t-1})$, but in this case, the second sum equals zero. i.e. $G_t(4mnp^b - y^2) = 0$. So $\widehat{F}(y) = g(1, p^t) G_t(4mnp^b - y^2) = 0$, implying $F(x) = 0$.

(e) Refer to [4], cf. [3, Section 14]. □

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